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Highlights

- We present several new formulations for the classical Capacitated Vehicle Routing Problem.
- We survey other previous flow-based formulations.
- We show that our new formulations have desired properties.
- We prove that the linear programming relaxations of the new formulations give better bounds.
- We analyze computational results to compare the known and new formulations.

Stronger Multi-Commodity Flow Formulations of the Capacitated Vehicle Routing Problem

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Abstract

The *Capacitated Vehicle Routing Problem* is a much-studied (and strongly $\mathcal{NP}$-hard) combinatorial optimization problem, for which many integer programming formulations have been proposed. We present two new *multi-commodity flow* (MCF) formulations, and show that they dominate all of the existing ones, in the sense that their continuous relaxations yield stronger lower bounds. Moreover, we show that the relaxations can be strengthened, in pseudo-polynomial time, in such a way that all of the so-called *knapsack large multistar* (KLM) inequalities are satisfied. The only other relaxation known to satisfy the KLM inequalities, based on set partitioning, is strongly $\mathcal{NP}$-hard to solve. Computational results demonstrate that the new MCF relaxations are significantly stronger than the previously known ones.

Keywords: Vehicle Routing, Multi-Commodity Flows, Integer Programming.

1. Introduction

Vehicle Routing Problems (VRPs) are classic problems in operational research and logistics, and have also received a great deal of attention from the combinatorial optimization community. A huge number of papers have been

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written on the theory and applications of VRPs, and on exact and heuristic solution methods for them (see, e.g., the edited volumes [5, 14, 27].)

This paper is concerned with the *Capacitated* VRP (CVRP), which Dantzig and Ramser [7] defined as follows. A fleet of identical vehicles, with limited capacity, is located at a depot. There are n customers that require service. Each customer has a known demand. The cost of travel between any pair of customers, or between any customer and the depot, is also known. The task is to find a minimum-cost collection of vehicle routes, each starting and ending at the depot, such that each customer is visited by exactly one vehicle, and no vehicle visits a set of customers whose total demand exceeds the vehicle capacity.

Letchford & Salazar [22] surveyed and compared several integer programming formulations of the CVRP. These included the so-called *two-* and *three-index* formulations, the *single-*, *two-* and *multi-commodity flow* formulations, and the *set partitioning* formulations. At present, the most successful exact algorithms for the CVRP are based on the two-index formulation (e.g., Lysgaard *et al.* [23]) or on set partitioning formulations (e.g., Fukasawa et al. [10], Baldacci *et al.* [3]).

One way to measure the strength of an alternative formulation is to project the feasible region of its continuous relaxation into the space of the natural (two-index) formulation. Gouveia [15] showed that, in the case of the single-commodity flow formulation, the projection satisfies a family of valid inequalities now known as *generalized large multistar* (GLM) inequalities. Letchford & Salazar [22] showed that the projection of the set partitioning formulation (with only elementary routes permitted) satisfies the so-called *knapsack large multistar* (KLM) inequalities, defined in [21]. The KLM inequalities include the GLM inequalities and the so-called *subtour elimination* (SE) inequalities as special cases. Unfortunately, the continuous relaxation of the set partitioning formulation is itself strongly $\mathcal{NP}$-hard to solve.

This paper has four main contributions. First, we show how to strengthen the two best multi-commodity flow (MCF) formulations, by adding only a polynomial number of additional constraints. Second, we show that the projections

of our two formulations satisfy the GLM and SE inequalities. Third, we show that the new formulations can be further strengthened, in pseudo-polynomial time, in such a way that all of the KLM inequalities are satisfied. (We remark that no polynomial or pseudo-polynomial time separation algorithm is known for the KLM inequalities themselves.) Finally, we present some computational results that demonstrate that the new MCF formulations are significantly stronger than the previously known ones.

As mentioned above, the current best algorithms for the CVRP are based on two-index or set partitioning formulations. The contribution of this paper may therefore appear to be only of theoretical interest. We would like to point out, however, that there exist variants of the CVRP for which it is natural, or even essential, to use additional commodity-flow variables. This includes, for example, the problem described in in [17], in which several distinct products have to be picked up and delivered at various locations, and the one described in [19], in which the cost of traversing an arc is an increasing function of vehicle load. Potentially, our results could be used to derive better formulations and algorithms for such problems.

The structure of the paper is as follows. The literature is reviewed in Section 2. The strengthened MCF formulations are presented and analysed in Section 3. The result on KLM inequalities is given in Section 4. Some computational results are given in Section 5, and some concluding remarks are made in Section 6.

Throughout the paper, we use the following notation. We have a complete directed graph G with node set $V = \{0, 1, \ldots, n\}$ and arc set A. Node 0 represents the depot, and nodes $1, \ldots, n$ represent customers. We sometimes write V_c for $V \setminus \{0\}$, the set of customer nodes. The (positive integer) demand of customer $i \in V_c$ is q_i. The (positive integer) vehicle capacity is Q. The (non-negative integer) cost of traversing arc $(i, j) \in A$ is c_{ij}. (Our approach can easily be adapted to the case of symmetric costs and/or the case in which the number of vehicles is restricted.)

2. Literature Review

As mentioned above, many formulations have been proposed for the CVRP. For brevity, we review only ones of relevance here. Subsections 2.1 to 2.4 cover *two-index vehicle flow*, *single-* and *two-commodity flow*, *multi-commodity flow* and *set partitioning* formulations, respectively.

2.1. The two-index vehicle flow formulation

Laporte & Nobert [20] presented what is now called the *two-index vehicle flow* formulation. For all $(i, j) \in A$, define a binary variable x_{ij}, taking the value 1 if and only if some vehicle travels from i to j. For any $S \subset V$, let $\delta^+(S)$ (respectively, $\delta^-(S)$) denote the set of arcs (i, j) with $i \in S, j \in V \setminus S$ (respectively, with $i \in V \setminus S, j \in S$). If $S = \{i\}$ then we will write $\delta^+(i)$ and $\delta^-(i)$ rather than $\delta^+(\{i\})$ and $\delta^-(\{i\})$, for brevity. Given some $F \subset A$, let $x(F)$ denote $\sum_{(i,j)\in F} x_{ij}$. Finally, for any set of customers $S \subset V_c$, let $q(S) = \sum_{i\in S} q_i$. Then the formulation is:

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} && && (1)\\
\text{s.t.} \quad & x(\delta^+(i)) = 1 && (i \in V_c) && (2)\\
& x(\delta^-(i)) = 1 && (i \in V_c) && (3)\\
& x(\delta^+(S)) \geq \lceil q(S)/Q \rceil && (S \subseteq V_c) && (4)\\
& x_{ij} \in \{0, 1\} && ((i, j) \in A). && (5)
\end{aligned}
$$

The *out-degree equations* (2) and the *in-degree equations* (3) ensure that vertices are visited exactly once. The constraints (4), called *rounded capacity* (RC) inequalities, prevent the existence of infeasible routes, and also have the side-effect of preventing subtours. Finally, (5) are the integrality conditions on the x-variables.

Several families of valid linear inequalities (cutting planes) have been developed for the two-index vehicle flow formulation (see Naddef & Rinaldi [25] for a survey). We will be interested in the following inequalities:

- The *fractional capacity* (FC) inequalities:

$$x(\delta^+(S)) \geq \frac{q(S)}{Q} \qquad (S \subseteq V_c). \tag{6}$$

- The *subtour elimination* (SE) inequalities:

$$x(\delta^+(S)) \geq 1 \qquad (S \subseteq V_c). \tag{7}$$

- The *generalized large multistar* (GLM) inequalities (see Gouveia [15]):

$$x(\delta^+(S)) \geq \frac{1}{Q} \sum_{i \in S} \big(q_i + \sum_{j \in V_c \setminus S} q_j (x_{ij} + x_{ji})\big) \qquad (S \subseteq V_c). \tag{8}$$

- The *knapsack large multistar* (KLM) inequalities (see Letchford *et al.* [21]):

$$x(\delta^+(S)) \geq \frac{1}{\beta} \sum_{i \in S} \big(\alpha_i + \sum_{j \in V_c \setminus S} \alpha_j (x_{ij} + x_{ji})\big) \qquad (S \subseteq V_c), \tag{9}$$

  where $\alpha \geq 0$ and $\beta > 0$ are such that the inequality $\sum_{i \in V_c} \alpha_i y_i \leq \beta$ is valid for the 0-1 knapsack polytope:

$$KP(Q, q) := \text{conv} \left\{ y \in \{0,1\}^n : \sum_{i \in V_c} q_i y_i \leq Q \right\}. \tag{10}$$

Obviously, the RC inequalities dominate the FC and SE inequalities, and the GLM inequalities dominate the FC inequalities. It is also not difficult to see that the KLM inequalities include the GLM and SE inequalities as special cases. In general, there are no other dominance relations.

2.2. Single- and two-commodity flow formulations

The first single-commodity flow formulation, that we call "`SCF1`", was presented by Gavish & Graves [12]. A continuous variable f_{ij} is defined for each $(i, j) \in A$, representing the total load (if any) carried along the arc (i, j). One then replaces constraints (4) in the two-index vehicle flow formulation with:

$$f(\delta^-(i)) - f(\delta^+(i)) = q_i \qquad (i \in V_c) \tag{11}$$

$$0 \leq f_{ij} \leq Q x_{ij} \qquad ((i, j) \in A). \tag{12}$$

The constraints (11) ensure that each customer i receives the demand of q_i. The constraints (12) are just bounds on the f variables.

Gavish [13] proposed to strengthen `SCF1` by replacing the bounds (12) with the stronger bounds

$$q_j x_{ij} \leq f_{ij} \leq (Q - q_i) x_{ij} \qquad ((i,j) \in A).$$

We call this strengthened formulation "`SCF2`".

Gouveia [15] used Hoffman's circulation theorem [18] to project the feasible regions of the LP relaxations of `SCF1` and `SCF2` into the space of the x variables. The projection of `SCF1` is given by the out-degree equations (2), the in-degree equations (3), the FC inequalities (6) and non-negativity. The projection of `SCF2`, as expected, is stronger, satisfying the GLM inequalities (8) in place of the FC inequalities.

In Baldacci *et al.* [2], a two-commodity flow formulation is presented for the case in which the costs c_{ij} are symmetric. Letchford & Salazar [22] show that, in this case, the LP relaxation of their formulation gives the same lower bound as that of `SCF2`.

2.3. Multi-commodity flow formulations

The first multi-commodity flow formulation, that we call "`MCF1a`", was presented in Garvin *et al.* [11]. A binary variable f_{ij}^k is defined for each $k \in V_c$ and each $(i,j) \in A$, taking the value 1 if and only if a vehicle traverses (i,j) on the way from the depot to k. The formulation is then obtained by replacing constraints (4) in the two-index vehicle flow formulation with:

$$f^k(\delta^+(0)) = f^k(\delta^-(k)) = 1 \qquad (k \in V_c) \qquad (13)$$

$$f^k(\delta^-(0)) = f^k(\delta^+(k)) = 0 \qquad (k \in V_c) \qquad (14)$$

$$f^k(\delta^+(i)) = f^k(\delta^-(i)) \qquad (k, i \in V_c : i \neq k) \qquad (15)$$

$$f_{ij}^k \geq 0 \qquad (k \in V_c, (i,j) \in A) \qquad (16)$$

$$\sum_{k \in V_c} q_k f_{ij}^k \leq Q x_{ij} \qquad ((i,j) \in A). \qquad (17)$$

The single-source multi-commodity flow theorem of Papernov [26] implies that the LP relaxations of `MCF1a` and `SCF1` are of equal strength.

Gavish [13] proposed an alternative formulation, that we call "`MCF1b`". It is obtained by replacing (17) with the following constraints:

$$\sum_{k \in V_c \setminus \{i\}} q_k f^k(\delta^+(i)) \leq Q - q_i \qquad (i \in V_c) \tag{18}$$

$$f^k_{ij} \leq x_{ij} \qquad (k \in V_c, (i,j) \in A). \tag{19}$$

It follows from the max-flow/min-cut theorem that, if the constraints (18) are dropped from `MCF1b`, then the projection into x-space is given by (2), (3), (7) and non-negativity. No similar projection result is known for `MCF1b` itself.

Letchford & Salazar [22] presented a different MCF formulation, with *two* commodities per customer. For each arc (i,j) and customer k, the variable f^k_{ij} is defined as before, but there is now also a binary variable g^k_{ij}, taking the value 1 if and only if a vehicle traverses (i,j) on the way from k to the depot. We then replace the constraints (4) in the two-index vehicle flow formulation with the constraints (13)–(16), together with:

$$g^k(\delta^+(k)) = g^k(\delta^-(0)) = 1 \qquad (k \in V_c) \tag{20}$$

$$g^k(\delta^-(k)) = g^k(\delta^+(0)) = 0 \qquad (k \in V_c) \tag{21}$$

$$g^k(\delta^+(i)) = g^k(\delta^-(i)) \qquad (k, i \in V_c : i \neq k) \tag{22}$$

$$g^k_{ij} \geq 0 \qquad (k \in V_c, (i,j) \in A) \tag{23}$$

$$f^k_{ij} + g^k_{ij} \leq x_{ij} \qquad (k \in V_c, (i,j) \in A) \tag{24}$$

$$\sum_{k \in V_c \setminus \{i\}} q_k \big(f^i(\delta^+(k)) + g^i(\delta^-(k)) \big) \leq Q - q_i \qquad (i \in V_c). \tag{25}$$

We will call this formulation "`MCF2a`". Note that the depot is either the source or the sink of every commodity. The above-mentioned result by Papernov [26] then implies that, if the constraints (25) are dropped from `MCF2a`, then the projection into x-space is again given by (2), (3), (7) and non-negativity. No similar projection result is known for `MCF2a` itself.

2.4. Set partitioning formulations

We will also need the following *set partitioning* (SP) formulation, due to Balinski & Quandt [4]. Let Ω denote the set of possible routes for a single vehicle, and let z_r for each $r \in \Omega$ be a binary variable taking the value 1 if and only if that route is used. Define the constant a_{ir} for each customer i and route r, taking the value 1 if i is served by r, and 0 otherwise. Finally let c_r denote the cost of route r. Then the SP formulation is:

$$\begin{aligned} \min \quad & \sum_{r \in \Omega} c_r z_r \\ \text{s.t.} \quad & \sum_{r \in \Omega} a_{ir} z_r = 1 && (i \in V_c) \\ & z_r \in \{0,1\} && (r \in \Omega). \end{aligned}$$

Since the number of variables in this formulation can be exponential in n, column generation is necessary. Unfortunately, the pricing subproblem is easily shown to be strongly $\mathcal{NP}$-hard. Agarwal *et al.* [1] solve it via integer programming. Foster & Ryan [9] noted that pricing becomes easier if one enlarges the set Ω by allowing routes in which the vehicle is permitted to visit customers more than once (now called *non-elementary* routes). Pricing can then be performed in pseudo-polynomial time, by dynamic programming. See, e.g., Martinelli *et al.* [24] for details.

Letchford and Salazar [22] prove the following:

- When elementary routes are used, the projection of the LP relaxation into x-space satisfies all KLM inequalities.
- Again, when elementary routes are used, the LP relaxation is at least as strong as those of all of the SCF and MCF formulations mentioned in the previous two subsections.
- If, however, non-elementary routes are permitted, then the only KLM inequalities which are satisfied by the projection are those in which $\alpha y \leq \beta$

is valid for the *general integer* knapsack polytope

$$\text{conv}\,\{y \in \mathbb{Z}^n_+ : \sum_{i \in V_c} q_i y_i \leq Q\}.$$

These less general KLM inequalities still include the GLM inequalities as a special case, but no longer include the SE inequalities.

3. Stronger Multicommodity Flow Formulations

In this section, we present four new multi-commodity flow formulations, each of which satisfies the SE inequalities (7) and GLM inequalities (8). The ones presented in Subsection 3.1 dominate `SCF1`, `SCF2`, `MCF1a` and `MCF1b`. The one presented in Subsection 3.2 also dominates `MCF2a`. The one presented in Subsection 3.3 is designed especially for instances with symmetric costs.

3.1. Strengthening MCF1a and MCF1b

In this subsection we will need the following lemma.

Lemma 1. *The LP relaxation of formulation* `MCF1b` *satisfies the equations*

$$f^j_{ij} = x_{ij} \qquad (j \in V_c, i \in V \setminus \{j\}). \tag{26}$$

Proof. Let $j \in V_c$ be fixed. From (3) we have $x(\delta^-(j)) = 1$, from (13) we have $f^j(\delta^-(j)) = 1$, and from (19) we have $f^j_{ij} \leq x_{ij}$ for all $i \in V \setminus \{j\}$. The only way for these to all hold simultaneously is for (26) to hold. □

The following proposition introduces a class of valid inequalities.

Proposition 1. *All (integer) solutions to formulation* `MCF1b` *satisfy the following inequalities:*

$$\sum_{k \in V_c \setminus \{i\}} q_k f^k_{ij} \leq (Q - q_i) x_{ij} \qquad ((i,j) \in A). \tag{27}$$

Proof. If the vehicle traverses the arc (i, j), then it must have already delivered a demand of q_i to customer i. □

Our first new formulation, which we call MCF1c, is obtained from MCF1b by replacing the constraints (18) with inequalities (27). The following two propositions state that MCF1c has some desirable properties.

Proposition 2. *The LP relaxation of MCF1c satisfies the SE inequalities (7) and the GLM inequalities (8).*

Proof. As mentioned in Subsection 2.3, the max-flow / min-cut theorem implies that the SE inequalities are satisfied. Now, for a given $S \subseteq V_c$ and a given commodity $k \in S$, the flow equations (13)–(15) imply that

$$f^k(\delta^-(S)) = f^k(\delta^+(S)) + 1.$$

Multiplying these equations by q_k and summing over all $k \in S$, we obtain

$$\sum_{k \in S} q_k f^k(\delta^-(S)) = \sum_{k \in S} q_k f^k(\delta^+(S)) + q(S). \tag{28}$$

Now, the constraints (27) for all $(i,j) \in \delta^-(S)$ imply that the left-hand side of (28) is no larger than

$$\sum_{(i,j) \in \delta^-(S)} (Q - q_i) x_{ij}.$$

On the other hand, the constraints (26) for all $(i,j) \in \delta^+(S)$, together with non-negativity, imply that the right-hand side of (28) is no smaller than

$$\sum_{(i,j) \in \delta^+(S)} q_j x_{ij} + q(S).$$

From this we deduce that the relaxation of MCF1d satisfies

$$\sum_{(i,j) \in \delta^-(S)} (Q - q_i) x_{ij} \geq \sum_{(i,j) \in \delta^+(S)} q_j x_{ij} + q(S),$$

which is equivalent to the GLM inequality (8) for the given S. □

Proposition 3. *The LP relaxation of MCF1c is stronger than that of SCF1, SCF2 and MCF1a, and at least as strong as that of MCF1b.*

Proof. The LP relaxation of `MCF1c` satisfies the constraints (2), (3) (7) and (8). The fact that it is stronger than the LP relaxations of `SCF1` and `SCF2` then follow from the result of Gouveia [15] mentioned in Subsection 2.2. It is also stronger than the LP relaxation of `MCF1a` since, as mentioned in Subsection 2.3, that relaxation is identical to the one of `SCF1`. To show that the LP relaxation of `MCF1c` is at least as strong as that of `MCF1b`, it suffices to show that it satisfies the inequalities (18). To this end, let $i \in V_c$ be fixed. Sum the inequalities (27) over all arcs entering i to obtain

$$\sum_{k \in V_c} q_k f^k(\delta^-(i)) \leq Q - \sum_{k \in V_c \setminus \{i\}} q_k x_{ki}.$$

Together with the equation (13) for $i = k$, this implies

$$\sum_{k \in V_c \setminus \{i\}} q_k f^k(\delta^-(i)) \leq Q - q_i - \sum_{k \in V_c \setminus \{i\}} q_k x_{ki}.$$

The equations (15) then imply

$$\sum_{k \in V_c \setminus \{i\}} q_k f^k(\delta^+(i)) \leq Q - q_i - \sum_{k \in V_c \setminus \{i\}} q_k x_{ki}.$$

This dominates the inequality (18) for the given i. □

Our computational results (Section 5) show that, in fact, the LP relaxation of `MCF1c` is stronger than that of `MCF1b`, and is also stronger than the relaxation in x-space defined by the out-degree equations (2), the in-degree equations (3), the SE inequalities (7), the GLM inequalities (8), and non-negativity.

Now, in the proof of Proposition 3, we showed that the inequalities (18) are redundant, being implied by the other constraints in `MCF1c`. Interestingly, however, they can be strengthened (lifted) to obtain a non-redundant family of inequalities. These stronger inequalities are presented in the following proposition.

Proposition 4. *All (integer) solutions to formulation `MCF1c` satisfy the following inequalities:*

$$\sum_{k \in V_c \setminus \{i\}} q_k \left(f^i(\delta^+(k)) + f^k(\delta^+(i)) \right) \leq Q - q_i \qquad (i \in V_c). \tag{29}$$

Proof. Consider the vehicle that services customer i. The two terms on the left-hand side represent the total demand delivered by the vehicle before arriving at customer i and after leaving customer i, respectively. □

Adding the inequalities (29) to `MCF1c`, we obtain a formulation that we call "`MCF1d`". (For ease of reference, we present `MCF1d` in its entirety in the Appendix.) The computational results in Section 5 show that the LP relaxation of `MCF1d` is stronger than that of `MCF1c`, which shows that the inequalities (29) are not implied by the other linear constraints in `MCF1d`.

3.2. Strengthening MCF2a

Now we turn our attention to `MCF2a`. We will show that `MCF2a` can be strengthened to obtain a formulation that also dominates `MCF1d`, the strongest of the formulations given in the previous subsection. Our starting point is the following proposition:

Proposition 5. *All (integer) solutions to formulation `MCF2a` satisfy the following constraints:*

$$f^k(\delta^+(i)) = g^i(\delta^-(k)) \qquad (k, i \in V_c : i \neq k). \tag{30}$$

Proof. The left-hand and right-hand sides are each equal to 1 if customer i is served before customer k on the same route, and 0 otherwise. □

The next step is to observe that, using Lemma 1, constraints (27) can be written as

$$\sum_{k \in V_c \setminus \{i,j\}} q_k f_{ij}^k \leq (Q - q_i - q_j) x_{ij} \qquad ((i,j) \in A),$$

The following proposition shows that these inequalities can then be strengthened, using the g-variables.

Proposition 6. *All (integer) solutions to formulation `MCF2a` satisfy the following constraints:*

$$\sum_{k \in V_c \setminus \{i,j\}} q_k \left(f_{ij}^k + g_{ij}^k \right) \leq (Q - q_i - q_j) x_{ij} \qquad ((i,j) \in A). \tag{31}$$

Proof. If a vehicle traverses the arc (i, j), then it must have already delivered a demand of q_i to customer i, and be about to deliver q_j to customer j. Therefore the total load of the vehicle dedicated to the other customers on the route cannot exceed $Q - q_i - q_j$. $\square$

We now show that the constraints (25) can be discarded when (30) and (31) are considered. To this end, we need the following lemma:

Lemma 2. *The LP relaxation of formulation* **MCF2a** *satisfies the equations*

$$f_{ij}^{j} = g_{ij}^{i} = x_{ij} \qquad (i, j \in V_c : i \neq j). \tag{32}$$

Proof. Similar to the proof of Lemma 1. $\square$

Proposition 7. *If the constraints (30) and (31) are added to* **MCF2a**, *then the constraints (25) become redundant.*

Proof. Use (21) and (32) to write (31) as:

$$\sum_{k \in V_c \setminus \{i\}} q_k \left(f_{ij}^{k} + g_{ij}^{k} \right) \leq (Q - q_i) x_{ij} \qquad ((i, j) \in A).$$

Sum these over all $j \in V \setminus \{i\}$, and use (2) to obtain:

$$\sum_{k \in V_c \setminus \{i\}} q_k \left(f^{k}(\delta^{+}(i)) + g^{k}(\delta^{+}(i)) \right) \leq Q - q_i \qquad (i \in V_c).$$

Reverse the roles of i and k, and use (15) and (30) to obtain (25). $\square$

Accordingly, we add constraints (30) and (31) to MCF2a, and delete the redundant constraints (25), to obtain what we call "MCF2b". (We present MCF2b in its entirety in the Appendix.) The following proposition shows that MCF2b is the strongest of all the MCF formulations considered so far.

Proposition 8. *Let (x^*, f^*, g^*) be a solution to the LP relaxation of* **MCF2b**. *Then (x^*, f^*) is a solution to the LP relaxation of* **MCF1d**.

Proof. The feasible region of the LP relaxation of `MCF1d` is defined by the constraints (2), (3), (13)–(16), (19), (27) and (29). Constraints (2), (3) and (13)–(16) are already present in `MCF2b`. Constraints (19) are dominated by (24). Constraints (27) are dominated by (31). Finally, the fact that constraints (29) are satisfied by (x^*, f^*) follows from Proposition 7 together with the fact that `MCF2b` contains the equations (30). □

Corollary 1. *The LP relaxation of formulation* `MCF2b` *satisfies the SE inequalities (7) and the GLM inequalities (8).*

Our computational results (Section 5) indicate that the lower bound given by `MCF2b` is signficantly stronger than the lower bounds given by either `MCF1d` or `MCF2a`. It is therefore the strongest known formulation based on commodity-flow variables.

3.3. An alternative formulation with fewer variables

To finish this section, we present a 'hybrid' formulation, which attempts to 'mimic' formulation `MCF2b`, but use only one commodity per customer, instead of two. We will show that, when a CVRP instance has symmetric costs, the lower bound from this hybrid formulation is just as good as the one from `MCF2b`.

The basic idea of the 'hybrid' formulation is to take the formulation `MCF2b` (given in the Appendix), and simplify it by using aggregated flow variables $\tilde{f}^k_{ij} = f^k_{ij} + g^k_{ji}$ for all $k \in V_c$ and $(i,j) \in A$. (Note that these aggregated variables have no natural physical interpretation.) The constraints in the resulting formulation are as follows:

$$
\begin{aligned}
\tilde{f}^k(\delta^+(0)) = \tilde{f}^k(\delta^-(k)) = 2 &\qquad (k \in V_c)\\
\tilde{f}^k(\delta^-(0)) = \tilde{f}^k(\delta^+(k)) = 0 &\qquad (k \in V_c)\\
\tilde{f}^k(\delta^+(i)) = \tilde{f}^k(\delta^-(i)) &\qquad (k, i \in V_c : i \neq k)\\
\tilde{f}^k_{ij} \geq 0 &\qquad (i, j, k \in V_c : i \neq j)\\
\tilde{f}^k_{ij} + \tilde{f}^k_{ji} \leq x_{ij} + x_{ji} &\qquad (i, j, k \in V_c : i < j)\\
\sum_{k \in V_c \setminus \{i,j\}} q_k(\tilde{f}^k_{ij} + \tilde{f}^k_{ji}) \leq (Q - q_i - q_j)(x_{ij} + x_{ji}) &\qquad (i, j \in V_c : i < j)
\end{aligned}
$$

$$\tilde{f}^k(\delta^+(i)) = \tilde{f}^i(\delta^+(k)) \qquad (k, i \in V_c : i \neq k).$$

We call this formulation "MCF3". Although MCF3 is weaker than MCF2b by construction, it does have some interesting properties. These are given in the following theorem and corollaries.

Theorem 1. *If the costs are symmetric, i.e. $c_{ij} = c_{ji}$ for all $(i,j) \in A$, then the lower bounds obtained by solving the LP relaxations of MCF2b and MCF3 are equal.*

Proof. Let $(x, \tilde{f})$ be any feasible solution to the LP relaxation of MCF3. We can construct a feasible solution to the LP relaxation of MCF2b, say $(\bar{x}, \bar{f}, \bar{g})$, using the following mapping:

$$\begin{aligned} \bar{x}_{ij} &= (x_{ij} + x_{ji})/2 \quad ((i,j) \in A) \\ \bar{f}^k_{ij} = \quad \bar{g}^k_{ij} &= (f^k_{ij} + f^k_{ji})/2 \quad ((i,j) \in A,\ k \in V_c). \end{aligned}$$

Moreover, if the costs are symmetric, then the cost of $(\bar{x}, \bar{f}, \bar{g})$ is the same as that of $(x, \tilde{f})$. This shows that, when costs are symmetric, the lower bound from MCF2b cannot be better than the one from MCF3. But it cannot be worse, since MCF3 is weaker than MCF2b by construction. □

Corollary 2. *The LP relaxation of MCF3 satisfies the SE inequalities (7) and the GLM inequalities (8).*

Proof. The in- and out-degree equations imply that $x(\delta^+(S)) = x(\delta^-(S))$ for all $S \subset V_c$. It follows that the SE inequalities can be written as

$$x(\delta^+(S)) + x(\delta^-(S)) \geq 2 \qquad (S \subseteq V_c),$$

and the GLM inequalities can be written as

$$x(\delta^+(S)) + x(\delta^-(S)) \geq \frac{2}{Q} \sum_{i \in S} \big(q_i + \sum_{j \in V_c \setminus S} q_j (x_{ij} + x_{ji})\big) \qquad (S \subseteq V_c).$$

In this form, the inequalities are symmetric (that is, for all $(i,j) \in A$, the coefficients of x_{ij} and x_{ji} are equal). The result then follows from Corollary 1 and Theorem 1. □

Corollary 3. *The LP relaxation of `MCF3` is stronger than those of `SCF1`, `SCF2` and `MCF1a`.*

Our computational results, given in Section 5, suggest that the LP relaxation of `MCF3` may be stronger than those of `MCF1b`, `MCF1c` and `MCF2a` as well. On the other hand, they show that there is no dominance relationship between the LP relaxations of `MCF3` and `MCF1d`.

4. Stronger Bounds in Pseudo-Polynomial Time

In this section we show that, if we are willing to sacrifice the property of having only a polynomial number of variables and constraints, then we can strengthen both `MCF1d` and `MCF2b` even further. In Subsection 4.1, we strengthen them by adding certain inequalities derived from facets of the 0-1 knapsack polytope, and show that the continuous relaxations of the resulting formulations satisfy all KLM inequalities. In Subsection 4.2, we present alternative strengthened formulations, of the same quality, which have additional variables, rather than constraints. We then show that the continuous relaxations of these latter formulations can be solved in pseudo-polynomial time, via column generation.

4.1. Valid inequalities from the knapsack polytope

The following lemma introduces exponentially-large families of valid inequalities that, in theory, could be used to strengthen further the formulations `MCF1d` and `MCF2b`.

Lemma 3. *Let $\alpha^T y \leq \beta$ be any valid inequality for the 0-1 knapsack polytope (10), with $\alpha \geq 0$ and $\beta > 0$. Then all (integer) solutions to formulation `MCF1d` satisfy the inequalities*

$$\sum_{k \in V_c \setminus \{i\}} \alpha_k f_{ij}^k \leq (\beta - \alpha_i) x_{ij} \qquad ((i,j) \in A), \tag{33}$$

and all (integer) solutions to formulation `MCF2b` satisfy the inequalities:

$$\sum_{k \in V_c \setminus \{i,j\}} \alpha_k \left(f_{ij}^k + g_{ij}^k \right) \leq (\beta - \alpha_i - \alpha_j) x_{ij} \qquad ((i,j) \in A). \tag{34}$$

Proof. Let $(\bar{x}, \bar{f})$ be a feasible integer solution to `MCF1d`. If $\bar{x}_{ij} = 0$, the inequality (33) holds triviallly. So, suppose that $\bar{x}_{ij} = 1$, i.e., a vehicle traverses the arc (i, j). Now consider the vector $\bar{y} \in \{0,1\}^n$ obtained by setting $\bar{y}_k$ to 1 if and only if commodity k is on the vehicle just before the vehicle arrives at node i. Then $\bar{y}$ must be an extreme point of the polytope (10), and therefore it satisfies $\alpha^T y \le \beta$. Moreover, $\bar{y}_i$ must equal 1, which implies

$$\sum_{k \in V_c \setminus \{i\}} \alpha_k \bar{y}_k \le (\beta - \alpha_i)\bar{x}_{ij}.$$

Now, for $k \in V_c \setminus \{i\}$, we must have $\bar{y}_k = \bar{f}^k_{ij}$. This implies that $\bar{f}$ satisfies the inequality (33).

The proof for `MCF2b` is similar. The only differences are that (i) we set $\bar{y}_k$ to 1 if and only if commodity k is on the vehicle when it leaves the depot and (ii) for $k \in V_c \setminus \{i, j\}$, we must have $\bar{y}_k = f^k_{ij} + g^k_{ij}$. □

We have the following result:

Theorem 2. *If all non-redundant inequalities of the form (33) are added to* `MCF1d`*, the continuous relaxation of the resulting formulation satisfies the KLM inequalities (9). The same holds if all non-redundant inequalities of the form (34) are added to* `MCF2b`*.*

Proof. To show that the relaxation of `MCF1d` with (33) added satisfies the KLM inequalities (9), just use the proof of Proposition 2 with α_i and β taking the place of q_i and Q, and (33) taking the place of (27). To show the same result for `MCF2b` with (34) added, just note that the inequalities (34) are stronger than the inequalities (33). □

4.2. Alternative formulations solvable by column generation

The models in the previous subsection have a drawback from a computational point of view. Not only are the constraints (33) and (34) exponential in number, but their coefficients are given only implicitly. To overcome this drawback, we present alternative formulations that are more explicit, but yield lower

bounds of the same quality. These formulations have an exponential number of variables rather than constraints.

Let P be the set of all possible loading patterns of a single vehicle. That is, each member of P is a subset of V_c whose total demand does not exceed Q. For each arc (i,j) and for each $p \in P$, let λ_{ij}^p be a binary variable, taking the value 1 if and only if a vehicle arrives at node i carrying loading pattern p, and then traverses arc (i,j). (This implies of course that i and j are in p). Then the model `MCF1d` is enlarged by adding the constraints

$$x_{ij} = \sum_{p \in P:\, i,j \in p} \lambda_{ij}^p \qquad ((i,j) \in A\colon i,j \in V_c) \tag{35}$$

$$f_{ij}^k = \sum_{p \in P:\, i,j,k \in p} \lambda_{ij}^p \qquad ((i,j) \in A\colon i,j \in V_c, k \in V_c \setminus \{i,j\}). \tag{36}$$

Now, for each arc (i,j) and for each $p \in P$, let μ_{ij}^p be a binary variable, taking the value 1 if and only if a vehicle that departed from the depot with loading pattern p goes on to traverse the arc (i,j). (This implies again that i and j are in p). Then the model `MCF2b` is enlarged by adding the constraints

$$x_{ij} = \sum_{p \in P:\, i,j \in p} \mu_{ij}^p \qquad ((i,j) \in A\colon i,j \in V_c) \tag{37}$$

$$f_{ij}^k + g_{ij}^k = \sum_{p \in P:\, i,j,k \in p} \mu_{ij}^p \qquad ((i,j) \in A\colon i,j \in V_c, k \in V_c \setminus \{i,j\}). \tag{38}$$

Observe that the members of P correspond to extreme points of the 0–1 knapsack polytope (10). Accordingly, we call the strengthened formulations "`MCF1K`" and "`MCF2K`", respectively. The following two theorems show that these formulations have two desirable properties.

Theorem 3. *The LP relaxations of formulations `MCF1K` and `MCF2K` satisfy the KLM inequalities (9).*

Proof. Let α and β be as in Lemma 3. From the definition of P it follows that $\sum_{k \in p} \alpha_k \le \beta$ for all $p \in P$. Therefore, for all $i \in V_c$ and all $p \in P$ such that $i \in p$, we have $\sum_{k \in p \setminus \{i\}} \alpha_k \le \beta - \alpha_i$. This implies

$$\sum_{p \in P:\, i,j \in p} \left(\sum_{k \in p \setminus \{i\}} \alpha_k \right) \lambda_{ij}^p \le (\beta - \alpha_i) \sum_{p \in P:\, i,j \in p} \lambda_{ij}^p$$

for all $(i,j) \in A$. Rearranging the left-hand side, and using (35) to simplify the right-hand side, we get

$$\sum_{k \in V_c \setminus \{i\}} \alpha_k \sum_{p \in P:\, i,j,k \in p} \lambda_{ij}^p \leq (\beta - \alpha_i) x_{ij} \qquad ((i,j) \in A\colon i,j \in V_c).$$

These constraints together with (36) then imply (33). In a similar way, constraints (37) and (38) imply (34). The result then follows from Theorem 2. □

Theorem 4. *The LP relaxations of* `MCF1K` *and* `MCF2K` *can be solved in pseudo-polynomial time.*

Proof. Suppose we have solved a restricted master problem associated with `MCF1K`, i.e., an LP obtained from `MCF1K` by relaxing the integrality condition and replacing P with a small (polynomial-sized) subset $P' \subset P$. Let ρ_{ij} and π_{ij}^k be the optimal dual prices for (35) and (36), respectively. For a given $(i,j) \in A$, there exists a column λ_{ij}^p with negative reduced cost if and only if

$$\max \left\{ \sum_{k \in V_c \setminus \{i,j\}} \pi_{ij}^k y_k : \sum_{k \in V_c \setminus \{i,j\}} q_k y_k \leq Q - q_i - q_j,\ y \in \{0,1\}^{n-2} \right\} > \rho_{ij}.$$

This is a 0-1 knapsack problem, that can be solved in $\mathcal{O}(nQ)$ time via dynamic programming [6]. The pricing problem for `MCF2K` is similar. Now note that pricing in an LP is equivalent to separation in the dual of the LP. The desired results then follow from the polynomial equivalence of separation and optimization [16]. □

We remark that this is the first time that an LP relaxation of the CVRP has been found that can be solved in pseudo-polynomial time, yet satisfies all of the KLM inequalities. (As mentioned in Subsection 2.4, the LP relaxation of the SP formulation with elementary routes satisfies the KLM inequalities, but is strongly $\mathcal{NP}$-hard to solve.) Moreover, no pseudo-polynomial separation algorithm is known for the KLM inequalities themselves.

4.3. Comparison with set partitioning

The strongest of our MCF formulations is `MCF2K`. A natural question is how the lower bound associated with `MCF2K` compares with the lower bound associated with the SP formulation given in Subsection 2.4. The answer to this question depends on whether or not the set Ω is permitted to contain columns that correspond to non-elementary routes. As mentioned at the end of Subsection 2.4, the LP relaxation with non-elementary routes permitted does not satisfy all KLM inequalities. Thus, the associated bound does not dominate the bound from `MCF2K`. The following theorem settles the question for the case in which only elementary routes are permitted:

Theorem 5. *If z^* is a solution to the LP relaxation of the SP formulation, then there exists a solution (x^*, f^*, g^*, μ^*) to the LP relaxation of `MCF2K` that has the same cost.*

Proof. For each arc $(i, j) \in A$ and each route $r \in \Omega$, let b_{ijr} be a binary constant which takes the value 1 if and only if route r traverses arc (i, j). Also, for any $(i, j) \in A$, any $r \in \Omega$ and any customer node $k \in V_c$, let d_{ijkr} (respectively, d'_{ijkr}) be a binary constant which takes the value 1 if and only if, in route r,

- vertex k is visited and
- the arc (i, j) is traversed on the way from the depot to k (respectively, on the way from k to the depot).

Finally, for any $(i, j) \in A$, $r \in \Omega$ and loading pattern $p \in P$, let t_{ijrp} be a binary constant which takes the value 1 if and only if a vehicle following route r departs from the depot with loading pattern p and then goes on to traverse arc (i, j). The desired quadruple (x^*, f^*, g^*, μ^*) is then created by setting:

- x^*_{ij} to $\sum_{r\in\Omega} b_{ijr} z^*_r$ for all $(i, j) \in A$;
- f^{*k}_{ij} to $\sum_{r\in\Omega} d_{ijkr} z^*_r$ for all $(i, j) \in A$ and all $k \in V_c$;
- g^{*k}_{ij} to $\sum_{r\in\Omega} d'_{ijkr} z^*_r$ for all $(i, j) \in A$ and all $k \in V_c$;

- μ_{ij}^{*p} to $\sum_{r\in\Omega} t_{ijrp} z_r^*$ for all $(i,j) \in A$ and $p \in P$.

This quadruple has the same cost as z^*, from the definition of c_r in the objective of the SP formulation. One can check that it also satisfies all of the linear constraints in the formulation MCF2K, i.e., the constraints (2), (3), (13)–(16), (20)–(24), (31), (30), (37) and (38). □

Theorem 5 implies that the lower bound from MCF2K is dominated by the lower bound from the SP formulation with elementary routes. We stress however that the former bound can be computed in pseudo-polynomial time, whereas the latter bound cannot (unless $\mathcal{P} = \mathcal{NP}$).

5. Computational Experiments

In this section, we report on some computational experiments. We stress from the outset that the goal of these experiments was not to solve large-scale CVRP instances to proven optimality, but rather to establish dominance relations between the lower bounds obtained when solving the LP relaxation of various formulations. (The development of a viable exact algorithm for the CVRP based on formulation MCF2K may be the topic of a future paper.) We found that, for this purpose, it was sufficient to use small instances with $n = 16$. The advantage of using these instances is that the RC inequalities (4) can be enumerated and added to the LP relaxation if desired. (No efficient separation procedure is known for the RC inequalities.)

We created both asymmetric and symmetric instances. In the asymmetric instances, the costs c_{ij} were randomly generated in $[0, 500]$. In the symmetric instances, the costs c_{ij} were obtained by computing the Euclidean distance between locations randomly distributed in the square $[0, 500] \times [0, 500]$, except the depot, which was located in the center of the square. We created instances with general demands (random integers in the range $[25, 33]$) and instances with only unit demands. For the instances with general demands, we considered $Q \in \{100, 150, 200\}$. For the instances with unit demands, we considered $Q \in \{4, 6, 8\}$. This led to twelve families of instances, and for each family we

Table 1: Average ratios for relaxations in x-space.

Type	SE	FC	GLM	RC	FC+SE	GLM+SE	GLM+RC
A-G-100	42.47	74.76	80.09	91.61	74.96	80.10	91.82
A-G-150	64.10	82.56	84.57	89.88	83.03	84.77	89.88
A-G-200	72.64	81.35	82.14	96.62	82.17	82.70	96.62
A-U-4	54.37	83.99	87.92	94.30	84.33	87.92	94.35
A-U-6	72.12	85.85	87.34	97.65	86.62	87.79	97.65
A-U-8	87.37	93.76	94.33	98.05	94.98	95.25	98.05
S-G-100	61.53	81.52	87.48	98.70	85.42	89.65	98.72
S-G-150	76.83	82.85	86.91	99.62	90.95	92.95	99.62
S-G-200	82.31	79.70	82.07	99.64	90.25	91.38	99.64
S-U-4	70.01	85.20	90.76	99.98	90.66	94.28	99.99
S-U-6	81.17	82.43	85.43	99.95	91.94	93.45	99.95
S-U-8	89.51	82.84	84.71	100.00	95.44	96.14	100.00

generated 20 instances. The instance generator and the formulations were implemented in FICO Xpress Mosel 3, and the source code is available to readers on request to the authors.

Table 1 gives the results for seven LP relaxations that only involve x-variables. The first column describes the instance type, and the remaining columns summarise the results that we obtained when adding various combinations of the SE, FC, GLM and RC inequalities to the LP relaxation consisting of (1)–(3) and non-negativity. Each figure is the average, over 20 instances, of the ratio between the lower bound and the optimum, expressed as a percentage. Table 2 gives analogous results for six LP relaxations that involve f-variables, Table 3 does the same for five relaxations that involve f- and g-variables, and Table 4 does the same for four SP relaxations, that involve x and z variables. Columns `nSP` and `eSP` refer to the SP relaxation with non-elementary and elementary routes, respectively. Columns `nSP+RC` and `eSP+RC` refer to the same relaxations strengthened with RC inequalities (4).

The main conclusion from the results in Table 1 is that the RC inequalities are the most important inequalities by far. Comparing Tables 1 and 2, we see that, as expected, `MCF1a` gives the same bound as the FC inequalities. We

Table 2: Average ratios for relaxations in (x, f)-space.

Type	MCF1a	MCF1b	MCF1c	MCF1d	MCF1K	MCF3
A-G-100	74.76	51.18	80.99	82.34	87.45	81.99
A-G-150	82.56	68.62	85.44	86.25	86.90	86.17
A-G-200	81.35	75.14	83.18	83.54	83.93	83.53
A-U-4	83.99	61.89	88.77	89.98	89.98	90.04
A-U-6	85.85	75.65	88.36	88.88	88.88	88.93
A-U-8	93.76	89.61	95.53	95.82	95.82	95.86
S-G-100	81.52	65.99	89.84	90.06	91.09	91.79
S-G-150	82.85	78.30	93.03	93.27	93.66	95.01
S-G-200	79.70	82.74	91.47	91.60	91.93	93.02
S-U-4	85.20	73.28	94.37	94.64	94.64	96.73
S-U-6	82.43	82.00	93.55	93.74	93.74	95.33
S-U-8	82.84	89.51	96.21	96.33	96.33	97.83

Table 3: Average ratios for relaxations in (x, f, g)-space.

Type	MCF2a	MCF2b	MCF2K	MCF2b+RC	MCF2K+RC
A-G-100	76.62	85.68	96.97	93.61	98.00
A-G-150	80.25	88.86	91.68	91.15	92.30
A-G-200	79.46	85.16	87.10	96.89	97.14
A-U-4	83.57	93.43	93.43	96.34	96.34
A-U-6	83.36	91.16	91.16	98.04	98.04
A-U-8	92.39	97.10	97.10	98.38	98.38
S-G-100	80.61	91.79	93.94	98.74	99.08
S-G-150	86.92	95.01	95.99	99.64	99.67
S-G-200	87.09	93.02	93.85	99.64	99.70
S-U-4	85.99	96.73	96.73	100.00	100.00
S-U-6	88.25	95.33	95.33	99.96	99.96
S-U-8	92.44	97.83	97.83	100.00	100.00

Table 4: Average ratios for relaxations in (x, z)-space.

Type	nSP	nSP+RC	eSP	eSP+RC
A-G-100	98.56	98.72	98.59	98.72
A-G-150	93.40	93.76	94.09	94.25
A-G-200	89.38	97.45	90.94	97.67
A-U-4	94.76	96.86	94.97	96.92
A-U-6	92.11	98.19	93.39	98.40
A-U-8	96.97	98.28	98.43	98.86
S-G-100	95.05	99.39	98.46	99.76
S-G-150	90.78	99.67	98.69	99.88
S-G-200	85.37	99.70	97.16	99.83
S-U-4	91.91	100.00	98.29	100.00
S-U-6	86.66	99.95	97.27	99.99
S-U-8	85.74	100.00	99.50	100.00

also see that the four new formulations give better bounds than `MCF1a` and `MCF1b`, and also better bounds than the SE and GLM inequalities combined. Note also that there is no dominance between `MCF1a` and `MCF1b`, but `MCF1b` tends to perform poorly. Moreover, as expected, `MCF1c`, `MCF1d` and `MCF1K` are of increasing strength. As for `MCF3`, we see that it appears to dominate `MCF1c`, but does not dominate `MCF1d`. Interestingly, `MCF1K` does not dominate `MCF3`, despite the fact that it satisfies all KLM inequalities and is weakly $\mathcal{NP}$-hard to compute. Also, `MCF1K` gives the same bound as `MCF1d` in the unit demand case.

Turning our attention to Table 3, we see that `MCF2a` consistently gives worse bounds than the SE and GLM inequalities combined. As expected, `MCF2b` dominates `MCF1d` and `MCF2a`, and gives the same bound as `MCF3` on the symmetric instances. Also as expected, `MCF2K` dominates `MCF1K` and `MCF2b`. In fact, it is significantly stronger than `MCF1K` in all cases. On the other hand, `MCF2K` gives the same bound as `MCF2b` in the unit demand case. Note also that using `MCF2b` or `MCF2K` in combination with the RC inequalities gives better results than using RC and GLM inequalities in combination. The difference is noticeable especially for the asymmetric instances.

Now consider Table 4. We see that, as expected, the SP relaxation with

non-elementary routes dominates the GLM relaxation. On the other hand, it is weaker even than MCF1c in some cases, despite being weakly $\mathcal{NP}$-hard to compute. Moreover, it neither dominates nor is dominated by MCF2K, which is also weakly $\mathcal{NP}$-hard to compute. As for the SP relaxation with elementary routes, which is strongly $\mathcal{NP}$-hard to compute, it dominates all of the MCF formulations. This is in accordance with Theorem 5. Finally, including RC inequalities makes a considerable difference in some cases.

To aid the reader, Figure 1 shows all known dominance relations between all relaxations considered. An arrow from one relaxation to another indicates that the latter is stronger than the former.

6. Conclusion

In this paper, we have surveyed the known CVRP formulations based on additional commodity-flow variables, and introduced several new multi-commodity flow formulations that are provably stronger than all of them, in both theory and practice. Of particular interest is the formulation that we have called MCF2b, which is of polynomial size and yields lower bounds that are significantly stronger than those obtained by all other known formulations of polynomial size.

A natural question for future research is whether one could devise a competitive exact algorithm for the CVRP based upon one of our new formulations, or some similar formulation. To do this, some kind of decomposition scheme might be needed. Another interesting research topic would be to completely characterise the projections into x-space of the feasible regions of the LP relaxations of MCF1b to MCF1d, MCF2a and MCF2b, and possibly devise efficient separation routines for them, so that the additional flow variables could be avoided. Finally, one could attempt to adapt our formulations to other similar vehicle routing problems, such as the CVRP with pickup-and-delivery [8], the multi-commodity TSP described in [17], or the VRP with load-dependent costs [19].

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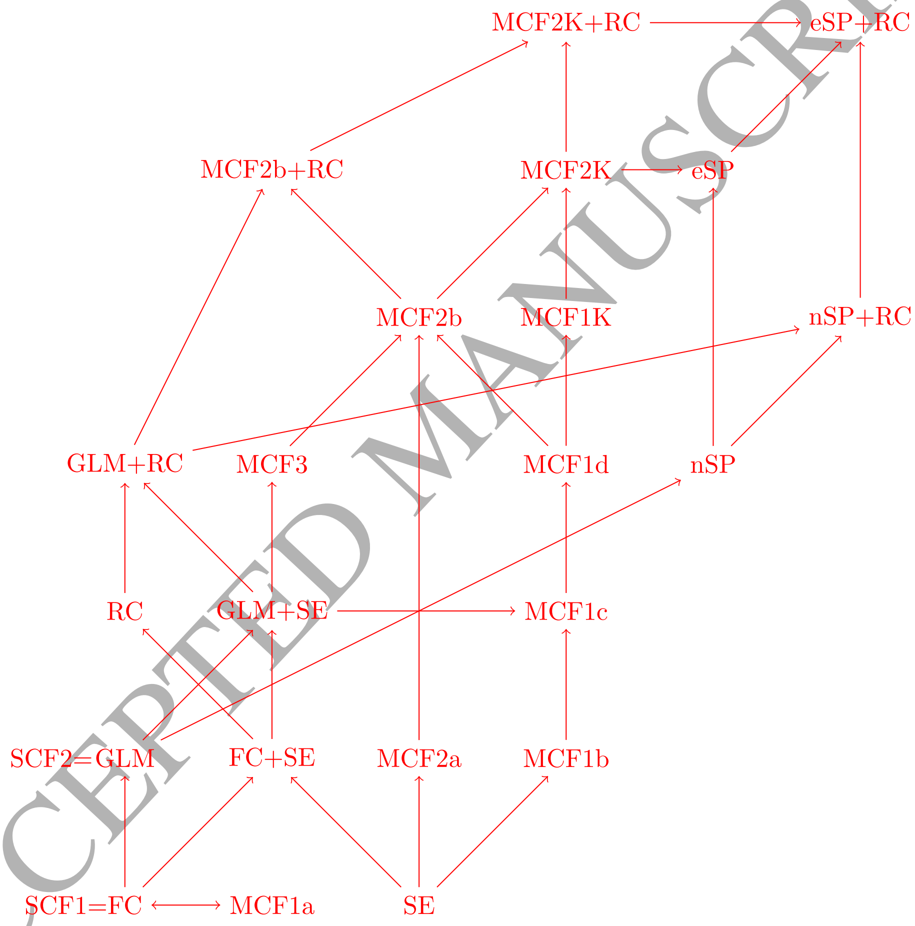
Figure 1: Dominance between formulations.



MTM2012-36163-C06-01.

References

[1] Agarwal, Y., Mathur, K., Salkin, H.M. (1989) A set-partitioning-based exact algorithm for the vehicle routing problem. Networks **19** 731-749 (1989)

[2] Baldacci, R., Hadjiconstantinou, E., Mingozzi, A.: An exact algorithm for the capacitated vehicle routing problem based on a two-commodity network flow formulation. Oper. Res. **52** 723–738 (2004)

[3] Baldacci, R., Christofides, N., Mingozzi, A.: An exact algorithm for the vehicle routing problem based on the set partitioning formulation with additional cuts. Math. Program. **115** 351–385 (2008)

[4] Balinski, M.L., Quandt, R.E.: On an integer program for a delivery problem. Oper. Res. **12** 300–304 (1964)

[5] Ball, M., Magnanti, T.L., Monma, C.L., Nemhauser, G.L. (eds.) *Network Routing*. Handbooks in Operations Research and Management Science, vol. 8. Amsterdam: North Holland (1995)

[6] Bellman, R.E.: *Dynamic Programming*. Princeton, NJ: Princeton University Press (1957)

[7] Dantzig, G.B., Ramser, R.H.: The truck dispatching problem. Mngt. Sci. **6**, 80–91 (1959)

[8] Desaulniers, G., Desrosiers, J., Erdmann, A., Solomon, M.M., Soumis, F.: VRP with pickup and delivery. In Toth, P., Vigo D. (eds.): *The Vehicle Routing Problem*. SIAM (2001)

[9] Foster, B.A., Ryan, D.M.: An integer programming approach to the vehicle scheduling problem. Oper. Res. Quart. **27** 367–384 (1976)

[10] Fukasawa, R., Lysgaard, J., de Aragão, M.P., Reis, M., Uchoa, E., Werneck, R.F.: Robust branch-and-cut-and-price for the capacitated vehicle routing problem. Math. Program. **106** 491–511 (2006)

[11] Garvin, W.W., Crandall, H.W., John, J.B., Spellman, R.A.: Applications of linear programming in the oil industry. Mgmt. Sci. **3** 407–430 (1957)

[12] Gavish, B., Graves, S.: The traveling salesman problem and related problems. Working Paper, Graduate School of Management, University of Rochester, New York (1979)

[13] Gavish, B.: The delivery problem: new cutting planes procedures. Presented at the TIMS XXVI conference, Copenhagen 1984.

[14] Golden, B.L., Raghavan, S., Wasil, E.A. (eds.): *The Vehicle Routing Problem: Latest Advances and New Challenges*. New York: Springer (2008)

[15] Gouveia, L. A result on projection for the vehicle routing problem. Eur. J. Oper. Res. **85** 610–624 (1995)

[16] Grötschel, M., Lovász, L., Schrijver, A.: The ellipsoid method and its consequences in combinatorial optimization. Combinatorica **1** 169–197 (1981)

[17] Hernández-Pérez, H., Salazar-González, J.J.: The multi-commodity one-to-one pickup-and-delivery traveling salesman problem. Eur. J. Oper. Res. **196** 987–995 (2009)

[18] Hoffman, A.J.: Some recent applications of the theory of linear inequalities to extremal combinatorial analysis. In Bellman, R.,Hall, M. (eds.) *Proceedings of Symposia in Applied Mathematics, Vol. X: Combinatorial Analysis*. Providence, RI: AMS (1960)

[19] Kara, I., Kara, B.Y., Yetis, M.K.: Energy minimizing vehicle routing problem. In Dress, A., Xu, Y., Zhu, B. (eds.) *Combinatorial Optimization and Applications*, pp. 62–71. Lecture Notes in Computer Science, vol. 4616, Springer: Berlin–Heidelberg (2007)

[20] Laporte, G., Nobert, Y.: A branch and bound algorithm for the capacitated vehicle routing problem. OR Spektrum **5** 77–85 (1983)

[21] Letchford, A.N., Eglese, R.W., Lysgaard, J.: Multistars, partial multistars and the capacitated vehicle routing problem. Math. Program. **94** 21–40 (2002)

[22] Letchford, A.N., Salazar-González, J.J.: Projection results for vehicle routing. Math. Program. **105** 251–274 (2006)

[23] Lysgaard, J., Letchford, A.N., Eglese, R.W.: A new branch-and-cut algorithm for the capacitated vehicle routing problem. Math. Program. **100** 423–445 (2004)

[24] Martinelli, R., Pecin, D., Poggi, M.: Efficient elementary and restricted non-elementary route pricing. Eur. J. Oper. Res. **239** 102–111 (2014)

[25] Naddef, D., Rinaldi, G.: Branch-and-cut algorithms for the capacitated VRP. In Toth, P., Vigo, D. (eds.): *The Vehicle Routing Problem*. SIAM (2001)

[26] Papernov, B.A.: On existence of multicommodity flows (in Russian). In Fridman, A.A. (ed.) *Studies in Discrete Optimization*. Moscow: Nauka (1976)

[27] Toth, P., Vigo, D. (eds.): *The Vehicle Routing Problem*. SIAM (2001)

Appendix

For ease of reference, we include here two of our multi-commodity flow formulations: `MCF1d`, which is the strongest known formulation of polynomial size that involves only x and f variables, and `MCF2b`, which is the strongest known formulation of polynomial size that involves x, f and g variables.

Formulation `MCF1d`:

$$
\begin{aligned}
\min \sum_{(i,j)\in A} c_{ij} x_{ij} & & \\
\text{s.t.} \quad x(\delta^+(i)) = x(\delta^-(i)) &= 1 & (i \in V_c) \\
x_{ij} &\in \{0,1\} & ((i,j) \in A) \\
f^k(\delta^+(0)) = f^k(\delta^-(k)) &= 1 & (k \in V_c) \\
f^k(\delta^-(0)) = f^k(\delta^+(k)) &= 0 & (k \in V_c) \\
f^k(\delta^+(i)) &= f^k(\delta^-(i)) & (k, i \in V_c : i \neq k) \\
0 \leq f^k_{ij} &\leq x_{ij} & (k \in V_c, (i,j) \in A) \\
\sum_{k \in V_c \setminus \{i,j\}} q_k f^k_{ij} &\leq (Q - q_i - q_j) x_{ij} & ((i,j) \in A) \\
\sum_{i \in V_c \setminus \{k\}} q_i \left(f^k(\delta^+(i)) + f^i(\delta^+(k)) \right) &\leq Q - q_k & (k \in V_c).
\end{aligned}
$$

Formulation `MCF2b`:

$$
\begin{aligned}
\min \sum_{(i,j)\in A} c_{ij} x_{ij} & & \\
\text{s.t.} \quad x(\delta^+(i)) = x(\delta^-(i)) &= 1 & (i \in V_c) \\
x_{ij} &\in \{0,1\} & ((i,j) \in A) \\
f^k(\delta^+(0)) = f^k(\delta^-(k)) = g^k(\delta^+(k)) = g^k(\delta^-(0)) &= 1 & (k \in V_c) \\
f^k(\delta^-(0)) = f^k(\delta^+(k)) = g^k(\delta^-(k)) = g^k(\delta^+(0)) &= 0 & (k \in V_c) \\
f^k(\delta^+(i)) = f^k(\delta^-(i)) = g^i(\delta^+(k)) &= g^i(\delta^-(k)) & (k, i \in V_c : i \neq k) \\
f^k_{ij}, g^k_{ij} \geq 0 \quad ; \quad f^k_{ij} + g^k_{ij} &\leq x_{ij} & (k \in V_c, (i,j) \in A) \\
\sum_{k \in V_c \setminus \{i,j\}} q_k \left(f^k_{ij} + g^k_{ij} \right) &\leq (Q - q_i - q_j) x_{ij} & ((i,j) \in A).
\end{aligned}
$$